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Representations of $SU(1,1)$ in non-commutative space generated by the Heisenberg algebra

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Abstract

$SU(1,1)$ is considered as the automorphism group of the Heisenberg algebra H. The basis in the Hilbert space K of functions on H on which the irreducible representations of the group are realized is explicitly constructed. From group theoretical considerations summation formulae for the product of two, three and four hypergeometric functions are derived.

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1. Introduction

Investigating the properties of manifolds by means of the symmetries they admit has a long history. Non-commutative geometries have become the subject of similar studies in recent decades. For example, there exists an extensive literature on the q-deformed groups $E_q(2)$ and $SU_q(2)$, which are the automorphism groups of the quantum plane $zz^* = qz^*z$ and the quantum sphere respectively [1]. Using group theoretical methods the invariant distance and the Green functions have also been written in these deformed spaces [2].

In recent work we started to analyse the non-commutative space $[z, z^*] = 1$ (i.e. the space generated by the Heisenberg algebra) by means of its automorphism groups: we considered $E(2)$ group transformations in z, z^* space; and constructed the basis (which are written in terms of the Kummer functions) in this space where the unitary irreducible representations of $E(2)$ are realized [3]. This analysis revealed a peculiar connection between the two-dimensional Euclidean group and the Kummer functions.

In this paper we continue to study the same non-commutative space $[z, z^*] = 1$, this time by means of the other admissible automorphism group $SU(1,1)$. Our basic motivation is to derive new summation formulae for the hypergeometric functions in general and Jacobi functions in particular. The procedure also gives new group theoretical interpretations to the already known formulae.

In section 2 we define $SU(1,1)$ in the Heisenberg algebra H and construct the unitary representations of the group in the Hilbert space X where H is realized.

In section 3 we classify the invariant subspaces in the space of the bounded functions on H where the irreducible representations of $SU(1,1)$ are realized.

In section 4 we show that in the Hilbert space K of the square integrable functions only the principal series is unitary. We construct the orthonormal basis in K, which can be written in terms of the Jacobi functions.

Section 5 is devoted to the derivation of summation formulae involving products of two, three and four hypergeometric functions.

In section 6 explicit examples are given.

2. Weyl representations of $SU(1,1)$

The one-dimensional Heisenberg algebra H is the three-dimensional vector space with the basis elements $\{z, z^*, 1\}$ and the bilinear antisymmetric product

$$[z, z^*] = 1. \tag{1}$$

The $*$-representation of H in the suitable dense subspace of the Hilbert space X with the complete orthonormal basis $\{|n\rangle\}$, $n = 0, 1, 2, \ldots$, is given by

$$z|n\rangle = \sqrt{n}|n-1\rangle \qquad z^*|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{2}$$

Let us represent the pseudo-unitary group $SU(1,1)$ in the vector space H:

$$g\begin{pmatrix} z \\ z^* \end{pmatrix} = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}\begin{pmatrix} z \\ z^* \end{pmatrix}. \tag{3}$$

Because

$$a\overline{a} - b\overline{b} = 1 \tag{4}$$

the transformations (3) preserve the commutation relation

$$[gz, gz^*] = [z, z^*]. \tag{5}$$

Therefore

$$gz = U(g)zU^{-1}(g) \qquad gz^* = U(g)z^*U^{-1}(g) \tag{6}$$

where $U(g)$ is the unitary representation of $SU(1,1)$ in X:

$$U(g_1)U(g_2) = U(g_1g_2) \qquad U^*(g) = U^{-1}(g) = U(g^{-1}). \tag{7}$$

The Cartan decomposition for the group reads

$$g = k(\phi)h(\alpha)k(\psi) \tag{8}$$

where

$$k(\psi) = \begin{pmatrix} \mathrm{e}^{\frac{\mathrm{i}\psi}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\frac{\psi}{2}} \end{pmatrix} \qquad h(\alpha) = \begin{pmatrix} \cosh\frac{\alpha}{2} & \sinh\frac{\alpha}{2} \\ \sinh\frac{\alpha}{2} & \cosh\frac{\alpha}{2} \end{pmatrix}. \tag{9}$$

For the subgroup $k(\psi)$ we have

$$U(k(\psi))|n\rangle = \mathrm{e}^{-\mathrm{i}\frac{n\psi}{2}}|n\rangle. \tag{10}$$

Let us choose the following realizations for z, z^* and X:

$$z = \frac{1}{\sqrt{2}}\left(x + \frac{\mathrm{d}}{\mathrm{d}x}\right) \qquad z^* = \frac{1}{\sqrt{2}}\left(x - \frac{\mathrm{d}}{\mathrm{d}x}\right) \tag{11}$$

$$\langle x|n\rangle = \Psi_n(x) \qquad \Psi_n(x) = \sqrt{\frac{\mathrm{e}^{-x^2}}{2^n n!\sqrt{\pi}}}H_n(x) \tag{12}$$

where H_n is the Hermite polynomial. From

$$h(\alpha)z = \frac{1}{\sqrt{2}}\left(x\mathrm{e}^{\frac{\alpha}{2}} + \mathrm{e}^{-\frac{\alpha}{2}}\frac{\mathrm{d}}{\mathrm{d}x}\right) \tag{13}$$

and

$$\int_{-\infty}^{\infty}\mathrm{d}x\,\overline{\Psi_m(x)}\Psi_n(x) = \delta_{nm} \tag{14}$$

we obtain

$$U(h(\alpha))\Psi_m(x) = \mathrm{e}^{\frac{\alpha}{4}}\Psi_m(\mathrm{e}^{\frac{\alpha}{2}}x). \tag{15}$$

Matrix elements of $U(h(\alpha))$ in the basis $\{|n\rangle\}$ read

$$U_{mn}(h) \equiv \langle m|U(h(\alpha))|n\rangle = \mathrm{e}^{\frac{\alpha}{4}}\int_{-\infty}^{\infty}\mathrm{d}x\,\overline{\Psi_m(x)}\Psi_n(\mathrm{e}^{\frac{\alpha}{2}}x). \tag{16}$$

Evaluating this integral we obtain

$$U_{mn}(h) = \frac{2^{\frac{m-n}{2}}}{(\frac{n-m}{2})!}\sqrt{\frac{n!\sinh^{n-m}\frac{\alpha}{2}}{m!\cosh^{n+m+1}\frac{\alpha}{2}}}\,F\left(-\frac{m}{2},\frac{1-m}{2};1+\frac{n-m}{2};-\sinh^2\frac{\alpha}{2}\right) \tag{17}$$

if $n \geqslant m$ and $n+m$ is even and

$$U_{mn}(h) = 0 \tag{18}$$

if $n+m$ is odd. For $m \geqslant n$ one has to replace m, n and α in the above formulae by n, m and $-\alpha$, respectively. We can express (17) through the Jacobi polynomial [5]:

$$U_{2m2n}(h) = \sqrt{\frac{m!\Gamma(n+1/2)}{n!\Gamma(m+1/2)}}\frac{\sinh^{n-m}\frac{\alpha}{2}}{\cosh^{n+m+1/2}\frac{\alpha}{2}}P_m^{(n-m,-\frac{1}{2}-n-m)}(\cosh\alpha) \tag{19}$$

$$U_{2m+12n+1}(h) = \sqrt{\frac{m!\Gamma(n+3/2)}{n!\Gamma(m+3/2)}}\frac{\sinh^{n-m}\frac{\alpha}{2}}{\cosh^{n+m+3/2}\frac{\alpha}{2}}P_m^{(n-m,-\frac{3}{2}-n-m)}(\cosh\alpha). \tag{20}$$

3. Irreducible representations of $SU(1,1)$ in H

The formula

$$T(g)F(z) = F(gz) \tag{21}$$

defines the representation of $SU(1,1)$ in the space K_0 of bounded operators in the Hilbert space X representable as the finite sums

$$F = \sum(f_n(\zeta)z^n + z^{*n}f_{-n}(\zeta)). \tag{22}$$

Here $f_n(\zeta)$ are functions of the self-adjoint operator $\zeta = z^*z$. Using (6) we can rewrite (21) in the form

$$T(g)F(z) = U(g)F(z)U^*(g). \tag{23}$$

With the one-parameter subgroups $g_1 = h(\epsilon)$, $g_2 = k(\frac{\pi}{2})h(\epsilon)k(-\frac{\pi}{2})$ and $g_3 = k(\epsilon)$ of $SU(1,1)$ we associate the linear operators $E_k : K_0 \to K_0$

$$E_k(F) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}(T(g_k)F - F) \tag{24}$$

with the limit being taken in the strong-operator topology. Inserting (23) into (24) we obtain (with $H_\pm = -E_1 \mp \mathrm{i}E_2$, $H = \mathrm{i}E_3$)

$$H_-(F) = \tfrac{1}{2}[F, z^2] \qquad H_+(F) = \tfrac{1}{2}[z^{*2}, F] \qquad H(F) = \tfrac{1}{2}[\zeta, F] \tag{25}$$

which implies the Lie algebra of $SU(1,1)$

$$[H_+, H_-] = 2H \qquad [H, H_\pm] = \pm H_\pm. \tag{26}$$

The irreducible representations labelled by the pair (τ, ϵ), $\tau \in C$ and $\epsilon = 0, \frac{1}{2}$, are given by the formulae [4]

$$H_- D_k^{(\tau,\epsilon)} = -(k + \tau + \epsilon) D_{k-1}^{(\tau,\epsilon)} \tag{27}$$

$$H_+ D_k^{(\tau,\epsilon)} = (k - \tau + \epsilon) D_{k+1}^{(\tau,\epsilon)} \tag{28}$$

$$H D_k^{(\tau,\epsilon)} = (k + \epsilon) D_k^{(\tau,\epsilon)}. \tag{29}$$

Equations (25) and (29) imply

$$D_k^{(\tau,\epsilon)} = z^{*2(k+\epsilon)} f_k^{(\tau,\epsilon)}(\zeta) \tag{30}$$

for $k \geqslant 0$ and

$$D_k^{(\tau,\epsilon)} = f_k^{(\tau,\epsilon)}(\zeta) z^{-2(k+\epsilon)} \tag{31}$$

for $k < 0$. By substituting (30) in (27) and (28) with

$$f_k^{(\tau,\epsilon)}(\zeta) = \sum_{n=0}^{\infty} \frac{(-)^n 2^{n+k+\epsilon}}{n!} C_{kn} z^{*n} z^n \tag{32}$$

we obtain the recurrence relations

$$nC_{kn-1} + \frac{k+\epsilon+\tau}{2k+2\epsilon+n-1} C_{k-1n} - (2k+2\epsilon+n) C_{kn} = 0 \tag{33}$$

$$C_{kn+1} - C_{kn+2} - (k+\epsilon-\tau) C_{k+1n} = 0 \tag{34}$$

which are solved by

$$C_{kn} = \frac{\Gamma(1+\tau+\epsilon+k+n)}{\Gamma(1+2\epsilon+2k+n)}. \tag{35}$$

Using

$$z^{*n} z^n = \zeta(\zeta-1)\cdots(\zeta-n+1) \tag{36}$$

for $k \geqslant 0$ we obtain

$$f_k^{(\tau,\epsilon)}(\zeta) = (-2)^{k'} \frac{\Gamma(1+\tau+k')}{\Gamma(1+2k')} F(-\zeta, 1+\tau+k'; 1+2k'; 2) \tag{37}$$

or

$$f_k^{(\tau,\epsilon)}(\zeta) = (-2)^{k'} (-)^{\zeta} \frac{\zeta!\,\Gamma(1+\tau+k')}{(\zeta+2k')!} P_\zeta^{(\tau-k'-\zeta,2k')}(3) \tag{38}$$

where $k' = k + \epsilon$. The functions $f_k^{(\tau,\epsilon)}$ for $k < 0$ are shown to be defined from the expression

$$f_k^{(\tau,\epsilon)}(\zeta) = f_{-k}^{(\tau,-\epsilon)}(\zeta). \tag{39}$$

From (27)–(29) we conclude that $SU(1,1)$ admits the following irreducible representations:

(i) $T_{(\tau,\epsilon)} : (\tau+\epsilon) \,\overline{\in}\, Z$
(ii) $T^{\pm}_{(\tau,\epsilon)} : (\tau+\epsilon) \in Z, \tau - \epsilon < 0$; that is, $\tau = -\frac{1}{2}, -1, -\frac{3}{2}, \ldots$
(iii) $T^{0}_{(\tau,\epsilon)} : (\tau+\epsilon) \in Z, \tau - \epsilon \geqslant 0$; that is, $\tau = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$.

The corresponding invariant subspaces are:

(i) $V_{(\tau,\epsilon)}$ generated by $\{D_k^{(\tau,\epsilon)}\}_{k=-\infty}^{\infty}$
(ii) $V^{+}_{(\tau,\epsilon)}$ and $V^{-}_{(\tau,\epsilon)}$ generated by $\{D_k^{(\tau,\epsilon)}\}_{k=-\infty}^{\tau-\epsilon}$ and $\{D_k^{(\tau,\epsilon)}\}_{k=-\tau-\epsilon}^{\infty}$
(iii) $V^{0}_{(\tau,\epsilon)}$ generated by $\{D_k^{(\tau,\epsilon)}\}_{k=-\tau-\epsilon}^{\tau-\epsilon}$.

4. Unitary irreducible representations of $SU(1,1)$ in H

We can define the norm in the subspace of K_0 with $f_n(\zeta)$ in (22), being the functions with finite support in $\mathrm{Spect}(\zeta) = \{0, 1, 2, \ldots\}$, as

$$||F|| = \sqrt{\mathrm{tr}(F^*F)}. \tag{40}$$

Completion of this subspace leads to the Hilbert space K of the square integrable functions in the linear space H with the scalar product

$$(F, G) = \mathrm{tr}(F^*G). \tag{41}$$

Using (23), the unitarity of $U(g)$ and the property of the trace, we conclude that the representation $T(g)$ in K is unitary. Equation (25) implies the real structure in the Lie algebra

$$H_\pm^* = -H_\mp \qquad H^* = H. \tag{42}$$

To investigate the unitarity of the irreducible representations in the Hilbert space K classified in the previous section we consider the orthogonality condition for the basis elements $D_k^{(\tau,\epsilon)}$. Using (2) and (30) we obtain

$$(D_k^{(\tau,\epsilon)}, D_m^{(\tau',\epsilon')}) = \delta_{mk}\delta_{\epsilon\epsilon'} \sum_{n=0}^{\infty} \frac{(n+2k+2\epsilon)!}{n!} \overline{f_k^{(\tau,\epsilon)}(-n)}\, f_k^{(\tau',\epsilon)}(-n). \tag{43}$$

Putting

$$s = 1 - \mathrm{e}^{-t} \qquad \lambda = 1 + 2(k+\epsilon) + \mu \tag{44}$$

in the formula [6]

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+\lambda)}{n!\Gamma(\lambda)} s^n F(-n, a; \lambda; 2) F(-n, b; \lambda; 2)$$
$$= (1-s)^{a+b-\lambda}(1+s)^{-a-b} F\left(a, b; \lambda; \frac{4s}{(1+s)^2}\right) \tag{45}$$

and taking first the limit $\mu \to +0$ and then $t \to \infty$ we obtain for $\tau = -\frac{1}{2} + \mathrm{i}\rho$, $\rho \in R$, the orthogonality relations

$$\left(D_k^{(-\frac{1}{2}+\mathrm{i}\rho,\epsilon)}, D_m^{(-\frac{1}{2}+\mathrm{i}\rho',\epsilon')}\right) = \delta_{mk}\delta_{\epsilon\epsilon'}\delta(\rho-\rho'). \tag{46}$$

In the derivation of the above relation we used

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{47}$$

and the representation

$$\lim_{t\to\infty} \frac{\mathrm{e}^{-\mathrm{i}zt}}{z + \mathrm{i}0} = -2\pi\mathrm{i}\delta(z) \tag{48}$$

for the Dirac delta function. For other values of τ there is no orthogonality condition. Thus in K only the representation $T_{(\tau,\epsilon)}$ with $\tau = -\frac{1}{2} + \mathrm{i}\rho$ of section 3, which is the principal series, is unitary.

5. The addition theorems

(i) Restriction of (21) on the subspace $V_{(\tau,\epsilon)}$ reads

$$T(g)D_k^{(\tau,\epsilon)} = \sum_{n=-\infty}^{\infty} t_{nk}^{(\tau,\epsilon)}(g)D_n^{(\tau,\epsilon)} \tag{49}$$

or

$$U(g)D_k^{(\tau,\epsilon)}U^*(g) = \sum_{n=-\infty}^{\infty} t_{nk}^{(\tau,\epsilon)}(g)D_n^{(\tau,\epsilon)} \tag{50}$$

where

$$t_{nk}^{(\tau,\epsilon)}(g) = \mathrm{e}^{-\mathrm{i}n'\phi-\mathrm{i}k'\psi}\frac{\Gamma(1+\tau-k')\sinh^{n'-k^pr}\frac{\alpha}{2}}{(n'-k')!\Gamma(1+\tau-n')\cosh^{k'+n'}\frac{\alpha}{2}}$$
$$\times F\left(-\tau-k', 1+\tau-k'; 1+n'-k'; -\sinh^2\frac{\alpha}{2}\right) \tag{51}$$

with $n' = n+\epsilon, k' = k+\epsilon$ the matrix elements of the irreducible representations which are valid for $n \geqslant k$. For $n < k$ one has to replace n and k on the right-hand side by $-n$ and $-k$ respectively.

(ii) Restriction of (21) on the subspaces $V_{(\tau,\epsilon)}^+$ and $V_{(\tau,\epsilon)}^-$ gives the following addition theorems:

$$U(g)D_k^{(\tau,\epsilon)}U^*(g) = \sum_{n=-\infty}^{\tau-\epsilon} t_{nk}^{(\tau,\epsilon)}(g)D_n^{(\tau,\epsilon)} \tag{52}$$

and

$$U(g)D_k^{(\tau,\epsilon)}U^*(g) = \sum_{n=-\tau-\epsilon}^{\infty} t_{nk}^{(\tau,\epsilon)}(g)D_n^{(\tau,\epsilon)}. \tag{53}$$

(iii) On the subspaces $V_{(\tau,\epsilon)}^0$ the addition theorem reads

$$U(g)D_k^{(\tau,\epsilon)}U^*(g) = \sum_{n=-\tau-\epsilon}^{\tau-\epsilon} t_{nk}^{(\tau,\epsilon)}(g)D_n^{(\tau,\epsilon)}. \tag{54}$$

Sandwiching both sides of (50) and (52)–(54) between the states $\langle l|$ and $|s\rangle$ we obtain

$$\sum_{m,t=0}^{\infty} U_{lm}(g)\overline{U_{st}(g)}(D_k^{(\tau,\epsilon)})_{mt} = \sum_n t_{nk}^{(\tau,\epsilon)}(g)(D_n^{(\tau,\epsilon)})_{ls} \tag{55}$$

where

$$(D_k^{(\tau,\epsilon)})_{mt} = (-)^{k'+t}2^{k'}\sqrt{\frac{t!}{m!}}\Gamma(1+\tau+k')P_t^{(\tau-k'-t,2k')}(3)\delta_{m,t+2k'} \tag{56}$$

for $k \geqslant 0$ and

$$(D_k^{(\tau,\epsilon)})_{mt} = (D_{-k}^{(\tau,-\epsilon)})_{tm} \tag{57}$$

for $k < 0$. Due to the Kronecker function in (56) the one summation from both sides of (55) can be lifted. Equation (55) gives the summation formula for the product of three hypergeometric functions (the one in $D_k^{(\tau,\epsilon)}$ is the Jacobi function with the constant argument (56)) through the matrix elements of irreducible representations of $SU(1,1)$.

Multiplying (50) and (52)–(54) by $U(g)$ from the right and sandwiching them between the states $\langle l|$ and $|s\rangle$ we obtain another summation formula

$$\sum_{m=0}^{\infty} U_{lm}(g)(D_k^{(\tau,\epsilon)})_{ms} = \sum_{m=0}^{\infty}\sum_{n} t_{nk}^{(\tau,\epsilon)}(g)(D_n^{(\tau,\epsilon)})_{lm}U_{ms}(g). \tag{58}$$

Multiplying (50) and (52)–(54) by $U^*(g)$ and $U(g)$ from the left and right respectively and sandwiching them between the states $\langle l|$ and $|s\rangle$ we obtain

$$(D_k^{(\tau,\epsilon)})_{ls} = \sum_{m,t=0}^{\infty}\sum_{n} t_{kn}^{(\tau,\epsilon)}(g)U_{ts}(g)\overline{U_{ml}(g)}(D_n^{(\tau,\epsilon)})_{mt} \tag{59}$$

which is the summation formula for the product of four hypergeometric functions.

6. Examples

It is clear that (55), (58) and (59) provide many summation formulae involving the product of hypergeometric functions. In this section, to give an idea about the explicit forms of these formulae, we give one simple example for each of them.

For the sake of simplicity we restrict our examples to the case (iii) with $k = \epsilon = 0$ and $g = h$.

(A) The explicit form of (55) with $l = s + 2r$ is

$$\sum_{t=0}^{\infty}(-)^t P_t^{(\tau-t,0)}(3)U_{s+2rt}(h)\overline{U_{st}(h)} = 0 \tag{60}$$

if $f \,\overline{\in}\, [-\tau, \tau]$ and

$$\sum_{t=0}^{\infty}(-)^t P_t^{(\tau-t,0)}(3)U_{s+2r,t}(h)\overline{U_{st}(h)} = \frac{(D_r^{(\tau,0)})_{s+2r,s}}{(\tau-|r|)!}P_\tau^{-|r|}(\cosh\alpha) \tag{61}$$

if $f \in [-\tau, \tau]$. Here we used

$$t_{n0}^{(\tau,0)}(h) = \frac{\tau!}{(\tau-|n|)!}P_\tau^{-|n|}(\cosh\alpha) \tag{62}$$

where $P_\tau^{\mu}(x)$ is the Legendre function [6]. For $s = 0$ and $r \geqslant 0$ the above expression becomes

$$\frac{1}{\sqrt{\cosh\frac{\alpha}{2}}}\sum_{t=0}^{\infty}\sqrt{\frac{\Gamma(t+\frac{1}{2})}{t!\sqrt{\pi}}}P_{2t}^{(\tau-2t,0)}(3)\tanh^t\left(\frac{\alpha}{2}\right)U_{2r,2t}(h) = \frac{2^r(\tau+r)!}{(2r)!(\tau-r)!}P_\tau^{-r}(\cosh\alpha) \tag{63}$$

which for $r = 0$ reads

$$\frac{1}{\sqrt{\pi}\cosh\frac{\alpha}{2}}\sum_{t=0}^{\infty}\frac{\Gamma(t+\frac{1}{2})}{t!}P_{2t}^{(\tau-2t,0)}(3)\tanh^{2t}\left(\frac{\alpha}{2}\right) = P_\tau(\cosh\alpha). \tag{64}$$

(B) The explicit form of (58) is

$$(D_0^{(\tau,0)})_{ss}U_{ls}(h) = \sum_{t=-\tau}^{\min(\tau,[l])}\frac{(D_n^{(\tau,0)})_{l,l-2n}\tau!}{(\tau-|n|)!}P_\tau^{-|n|}(\cosh\alpha)U_{l-2n,s}(h) \tag{65}$$

where $[l]$ is $\frac{l}{2}$ if l is even and $\frac{l-1}{2}$ if l is odd. For $s, l = 0$ we have

$$1 = \sum_{n=0}^{\tau}\frac{(-)^n!}{(n!)^2}\frac{(\tau+n)!}{(\tau-n)!}\tanh^{2n}\frac{\alpha}{2}F\left(-\tau, 1+\tau; 1+n; -\sinh^2\frac{\alpha}{2}\right). \tag{66}$$

(C) The explicit form of (59) is

$$\sum_{m=\max(0,2n)}^{\infty}\sum_{n=-\tau}^{\tau}\frac{(D_n^{(\tau,0)})_{m,m-2n}\tau!}{(D_0^{(\tau,0)})_{ss}(\tau-|n|)!}P_\tau^{-|n|}(\cosh\alpha)U_{m-2ns}(h)\overline{U_{ml}(h)}=\delta_{sl} \tag{67}$$

which for $\tau = 0$ defines the unitarity condition for $U(h)$ or the summation formula for the product of two Jacobi functions:

$$\sum_{m=0}^{\infty}U_{ms}(h)\overline{U_{ml}(h)}=\delta_{sl}. \tag{68}$$

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